QCD Sum Rules: Bridging the Gap between Short and Large Distances

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Abstract

I discuss aspects of the QCD sum rule method which attracted theorists' attention in earnest at a relatively late stage and are not yet fully solved. At first I briefly review such general topics as the structure of the operator product expansion in QCD and intrinsic limitations of the quark-hadron duality concept. In the second part I comment on holographic constructions — a focus of the current efforts to say something new on QCD at strong coupling. Of particular interest to me is the recent derivation of the vacuum magnetic susceptibility due to Son and Yamamoto. Remarkably, their result is the same as that obtained previously by Vainshtein in the field-theoretic framework. For reasons which I do not understand at the moment, the Vainshtein formula, unexpectedly, is not successfull numerically.

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1. Introduction

I was asked to open this special session devoted to the QCD sum rules (sometimes referred to as the SVZ sum rules) with a brief outline of the current status of this method in various applications to hadronic physics and, perhaps, some historical remarks. Instead, I decided to do something else. I will completely skip the second part since quite recently I published a paper [1] where the reader can find both the description of the evolution of the method and relevant anecdotes (see also [2]). I will not dwell on various (quite fruitful) recent applications which will be (hopefully) covered by other speakers. Instead, I will focus on some "afterthoughts" of a general nature, some issues which were not explored (or not fully explored) in due time, when the SVZ sum rules were in the making. They came into the limelight in the last ten years or so. Three main topics are

- The structure of the operator product expansion (OPE) in QCD;

- The limitations of the quark-hadron duality;

and, finally,

- Fast-forward in the past (some remarks on AdS/QCD).

Let me first recall that the basic concept of the SVZ sum rules, its foundation, is the following idea:

The QCD vacuum structure is complicated and not yet fully analytically understood despite significant progress, especially after the advent of supersymmetry in this range of questions [3]. For limited purposes one can try to represent the QCD vacuum by a few vacuum expectation values (VEVs) of local operators intended to theoretically approximate various correlation functions in an intermediate domain of distances — between short and asymptotically large. The gluon and quark condensates are most important, but one is welcome to add a few others. There are many reasons not to add too many, though. One of them, as I will discuss later, is the fact that OPE in QCD presents an asymptotic expansion. This is in contradistinction with OPE in a number of conformal field theories (with no intrinsic scale) in which it is believed to be better convergent or just convergent in the coordinate space. (Is it? A good question for a serious reflection ...).
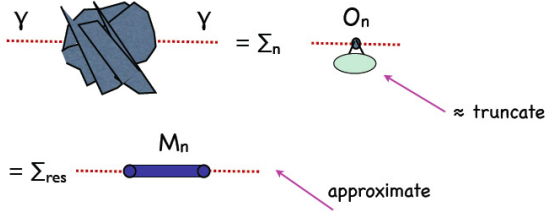
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Figure 1: Graphic representation of the SVZ sum rules in the correlation function $\langle \bar{\psi} \gamma^{\mu} \psi(x), \bar{\psi} \gamma^{\nu} \psi(0) \rangle$.

The above condensate expansion in the intermediate domain of distances can be matched by the sum over hadronic states, as in Fig. 1. The sum is infinite, but one can enhance the contribution from the lowest-lying states through Borelization.

2. Operator Product Expansion

The theoretical basis of any calculation within the SVZ method is the operator product expansion. It allows one to consistently define and build the (truncated) condensate series for any amplitude of interest in the Euclidean domain. The physical picture lying behind OPE is a consistent separation of short and large distance contributions. The latter are then represented by the vacuum condensates while the former are accounted for in the coefficient functions. Somewhat symbolically, the Fourier-transformed correlation functions can be represented as

$$D(q^2) = \sum_n C_n(q; \mu) \langle O_n(\mu) \rangle \tag{1}$$

where $D(q^2)$ is, say, a two-point function, and the normalization point μ is indicated explicitly. The sum in Eq. (1) runs over the Lorentz and gauge invariant operators built from the gluon and quark fields. The operator of the lowest (zero) dimension is the unit operator **I**, followed by the gluon condensate $G_{\mu\nu}^2$, of dimension four. The four-quark condensate gives an example of dimension-six operators. The OPE in (1) is, in a sense, a book-keeping procedure.

In this form Wilson designed it for theories with a UV fixed point at $\alpha \neq 0$ [4] (Wilson considered OPE in the coordinate space. The same was done by Polyakov in unpublished lectures on this topic that circulated about this time. In this case all x dependence is encoded in the set of coefficients C_n.) Theories with the UV fixed point at $\alpha \neq 0$ are conformal at short distances, with

the power-like approach to conformality. In this case it is most natural *not* to introduce a special normalization point μ. Instead, it is implicitly assumed to coincide with the external momentum q or distance x. OPE was thought of as the expansion in singularities at $x \to 0$ or $q \to \infty$.

The OPE (fusion rules) in the above form are fitted for the conformal theories (Polyakov et.al.) where dynamics at all scales is the same. In two dimensions the fusion rules are powerful enough to fully solve some CFTs [5]. In four-dimensional $\mathcal{N} = 4$ super-Yang–Mills theories conformal symmetry leads to miracles.

Alas ... our world is far less perfect.

- In QCD:

1) The UV fixed point is at $\alpha_s = 0$. Approach to the asymptotic limit is very slow — only logarithmic. Most operators cannot be defined as absolutely local. Anomalous dimensions are also logarithmic.

2) A dynamical scale of distances Λ^{-1} is generated through dimensional transmutation. Interactions on the opposite sides of Λ^{-1} are drastically different. Near and below Λ^{-1} perturbative calculations are inapplicable.

3) Without introduction of a normalization point μ, a sliding boundary between the two domains — short vs. large distances — construction of OPE is meaningless.

As a result, the Wilson–Polyakov original formula was not implemented as far as subleading power corrections are concerned. An appropriate modification of OPE was needed, fit for a very special QCD environment, with its logarithmic approach to the asymptotic limit. At the same time, the general Wilson's renormalization-group ideas are perfect! We did not change them in a conceptual way. Rather, we engineered its implementation most appropriate for QCD. Wilson's idea was finally fully adapted to the QCD environment in [6], with quite a number of successful forays beginning from 1978, in various SVZ papers.

Thus, Wilson's OPE was redesigned, with an important technical addition: we understood that power-suppressed terms of OPE need not be necessary discarded compared to logarithmically-suppressed terms, even if one cannot sum up the entire logarithmic series. They need not be smaller. I remember that implanting this idea in the minds of more formal theorists was a difficult task. Frankly speaking, even now a formal theoretical justification for this procedure is lacking. But it works!

- It was realized that the degree of locality of the operators $O_n(0)$ is regulated by an external parameter μ. Even if the fusion operators A and B have vanishing

anomalous dimensions (for instance, conserved vector currents), the coefficients $C_n^{AB}(x)$ and $O_n(0)$ depend on μ, in particular, through logarithms of μ, although the overall combination

$$\sum_n C_n^{AB}(x) O_n(0)$$

is μ independent! In practice, to make the SVZ method work for low-lying hadronic states one should be able to sail between Scylla and Charybdis of contradictory requirements on μ. If μ is too high or two low, one looses the predictive power! The value of μ is to be carefully balanced. I won't go into details here, referring to [7], and asserting that the necessary balance can be achieved in the so-called practical version of OPE. This is a fortuitous special feature of QCD which is not necessarily shared by simpler theories used to model various aspects of QCD. An example of such a simple theoretical laboratory is given by two-dimensional $CP(N-1)$ model [8]. Unlike the situation in $CP(N-1)$, in QCD, if μ is judiciously chosen, the coefficients $C_n^{AB}(0)$ are *mostly* determined by perturbation theory while the condensates $\langle O_n(0) \rangle$ are *mostly* mostly nonperturbative. We are lucky.

- The OPE expansion runs in powers of $1/Q^2$ and $\ln Q^2$.

- The OPE expansion is asymptotic at best. The fact that the condensate series is factorially divergent in high orders is rather obvious from the analytic structure of the polarization operator $D(Q^2)$. In a nut shell, since the cut in $D(Q^2)$ runs all the way to infinity along the positive real semi-axis of q^2, the $1/Q^2$ expansion cannot be convergent. The actual argument is somewhat more subtle [9] but the final conclusion is perfectly transparent. It is intuitively clear that the high-order tail of the (divergent) power series gives rise to exponentially small corrections $\sim \exp(-Q^\sigma)$ (where σ is some critical index) in the Euclidean domain which oscillate and suppressed by powers of energy (or other appropriate parameters) in the Minkowski domain.

The numerical value of σ is correlated with the rate of divergence of high orders in the power series. This is explained in great detail in Section 8 of [10]. At the moment very little is known about this rate from first principles, if at all. The best we can do is to rely on toy models [10]. The simplest example is provided by instantons. One has to fix the size of the instanton ρ by hand, $\rho = \rho_0$. Then the fixed-size instanton contribution in the Euclidean domain is indeed exponential, $O(\exp(-Q\rho_0))$. The exponential factor is the price we pay for transmitting the large momentum Q through a soft field configuration whose characteristic frequencies

are or the order of ρ_0^{-1}. Being analytically continued in the Minkowski domain, the imaginary part of the instanton contribution on the cut oscillates and is only power suppressed.

A very similar situation takes place in an alternative construction, the so-called resonance model worked out in Ref. [10]. Both models are on the market for quite a time, but — alas — there were essentially no recent advances in this direction. Any fresh ideas on possible better or more compelling models that would lie closer to first principles are most welcome!

Now, let me recall that not only the condensate series, but the α_s series *per se* are asymptotic. Some of the high-order divergences in the α_s series can be absorbed in the condensates normalized at μ (e.g. infrared renormalons). Others still can show up as nonperturbative terms in the expansion coefficients $C_n^{AB}(0)$, which, unfortunately, continue to be rather poorly controllable till present. Our knowledge of these terms is semi-empiric. They are known to be numerically small and neglectable in a variety of channels (but not in all! see [11] where exceptional channels are discussed at length) under a judicial choice of the normalization point μ.

Concluding this part of the talk I can summarize the general achievements in understanding OPE in QCD as follows: It became clear that the overall structure of this expansion is highly complicated – much more complicated than, say, in the (exactly solved part of) $\mathcal{N} = 2$ super-Yang–Mills [3]. Even in the Euclidean domain this expansion is asymptotic in various ways and includes contributions of different nature. It is quite timely to try to categorize those contributions which go beyond the truncated condensate series containing just a few terms. This is difficult. Is it doable?

3. Quark-hadron duality

This section could have been entitled "Parametrizing ignorance." If theoretical calculation is hard enough in the Euclidean domain, the problem is immensely exacerbated upon transition to the Minkowski kinematics which is necessary for two related reasons: (a) estimate of the continuum in the SVZ sum rules (not so crucial in the majority of instances); (b) predictions for highly excited states (crucial in a number of problems, such as restoration/nonrestoration of chiral symmetry in high excitations [12], hadronic widths of τ and similar, Regge trajectories at large N and so on).

If one could calculate $D(Q^2)$ in the Euclidean domain *exactly*, one could analytically continue the result to the Minkowski domain, and then take the imaginary part. The spectral density $\rho(s)_{\text{theor}}$ obtained in this way would

present the *exact* theoretical prediction for the measurable hadronic cross section. *There would be no need for duality.*

In practice, our calculation of $D(Q^2)$ is approximate, for many reasons. First, nobody is able to calculate the infinite $\alpha_s(Q^2)$ series for the coefficient functions, let alone the infinite condensate series. Both have to be truncated at some finite order. A few lowest-dimension condensates that can be captured, are known approximately. The best we can do is analytically continue the *truncated* theoretical expression, term by term, from positive to negative Q^2. For each term in the expansion the imaginary part at positive q^2 (negative Q^2) is well-defined. We assemble them together and declare the corresponding $\rho(s)_{\text{theor}}$ to be dual to the hadronic cross section $\rho(s)_{\text{exp}}$. In the given context "dual" means equal.

Let me elucidate this point in more detail. Assume that $D(Q^2)$ is calculated through α_s^2 and $1/Q^4$, while the terms α_s^3 and $1/Q^6$ (with possible logarithms) are dropped. Then the theoretical quark-gluon spectral density, obtained as described above, is expected to coincide with $\rho(s)_{\text{exp}}$, with the uncertainty of order $O[(\alpha_s(s))^3]$ and $O(1/s^3)$. The uncertainty in the theoretical prediction of this order of magnitude is *natural* since terms of this order are neglected in the theoretical calculation of $D(Q^2)$. If the coincidence in this corridor does take place, we say that the quark-gluon prediction is dual to the hadronic spectral density. If there are deviations going *beyond the natural uncertainty*, we call them violations of duality. Needless to say, once our calculation of $D(Q^2)$ becomes more precise, the definition of the "natural uncertainty" in $\rho(s)_{\text{theor}}$ changes accordingly.

This is the most clear-cut definition I can suggest. From the formal standpoint, it connects the duality violation issue with that of analytic continuation from the Euclidean to Minkowski domain. Negligibly small corrections (legitimately) omitted in the Euclidean calculations may and do get enhanced in Minkowski. Exponentially small terms at positive Q^2 become oscillating and only power-suppressed (at best) at positive q^2.

These oscillating terms defy the hierarchical ordering and can be referred to as "duality violating." The physical picture behind them is as follows. The duality violations are due to (i) rare atypical events, when the basic high-momentum quark transition occurs at large rather than short distances; (ii) residual interactions occurring at large distances between the quarks produced at short distances. In the first case appropriate (Euclidean) correlation functions develop singularities at finite x^2, while the second mechanism is correlated with

the $x^2 \to \infty$ behavior.

In both cases the duality violating component follows the same pattern: *exponential in Euclidean and oscillating in Minkowski*. Three distinct regimes were identified and considered in the literature so far:

- (i) Finite-distance singularities

$$s^{-\kappa/2}\,\sin(\sqrt{s})\,; \tag{2}$$

- (ii) Infinite-distance singularities ($N_c = \infty$)

$$s^{-\eta/2}\,\sin(s)\,; \tag{3}$$

- (iii) Infinite-distance singularities (N_c large but finite, $s \to \infty$)

$$\exp(-\alpha s)\sin(s), \quad \alpha = O\!\left(\frac{1}{N_c}\right) \ll 1\,. \tag{4}$$

These regimes are not mutually exclusive – in concrete processes one may expect the duality violating component to be a combination of (i) and (ii), or (i) and (iii). From the theoretical standpoint it is quite difficult to consistently define the duality violating component of the type (3). An operational definition I might suggest is as follows: Start from the limit $N_c = \infty$ and identify the component of the type (2). Follow its evolution as N_c becomes large but finite.

Now a few words about global-versus-local dualities misconceptions are in order. (This topic is also related to the issue of ordering of the limits $N_c \to \infty$, E fixed, or the other way around, $E \to \infty$, N_c fixed. These two limits are not interchangeable. As a result, $D(Q, N_c)$ must contain nonanalytic terms in $1/Q$ and $1/N_c$.)

Usually by local duality people mean point-by-point comparison of $\rho(s)_{\text{theor}}$ and $\rho(s)_{\text{exp}}$, while global duality compares the spectral densities $\overline{\rho(s)}$ averaged over some *ad hoc* interval of s, with an *ad hoc* weight function $w(s)$,

$$\int_{s_1}^{s_2} ds\, w(s)\,\rho(s)_{\text{theor}} \approx \int_{s_1}^{s_2} ds\, w(s)\,\rho(s)_{\text{exp}}\,.$$

Many authors believe that global duality defined in this way has a more solid status than local duality. Some authors go so far as to say that while global duality is certainly valid at high energies, this is not necessarily the case for local duality. This became a routine statement in the literature. Well, routine does not mean correct.

In fact, both procedures have exactly the same theoretical status. The point-by-point comparison, as well as the comparison of $\overline{\rho(s)}$'s (with an *ad hoc* weight function), must be considered as distinct versions of local

duality. The distinction between the "local" quantities, such as $R(e^+e^-)$ at a certain value of s and the integrals of the type involved, say, in R_τ is quantitative rather than qualitative. In both quantities there is a duality violating component, the only distinction is in the concrete values of the indices of the power fall-off in Eqs. (2) and (3), say, 3 vs. 6. Thus, the averaging over s, as in R_τ, makes the duality violation somewhat more suppressed, but this is still something which we totally miss in the *practical* version of OPE. In addition, at the moment these indices are model dependent. There is *no way* to reliably determine the value of duality violation, be it point by point as in $R(e^+e^-)$ or integrally, as in R_τ, from the analysis of the practical OPE *per se*.

The genuine global duality applies only to special integrals which can be *directly expressed through the Euclidean quantities*. For instance, if the integration interval extends from zero to infinity, and the weight function is exponential, the integral

$$\int_0^\infty ds \exp\{-s/M^2\}\rho(s)\,,\tag{5}$$

reduces to the Borel transform of $D(Q^2)$ in the Euclidean domain (i.e. at positive Q^2). For such quantities, duality can*not* be violated, by definition. [1]

In the limit $E \to \infty$, N_c fixed, dynamics *per se* "globalizes" duality, since the resonance widths are switched on and produce a smearing of the spectral density. This dynamical smearing may increase the indices κ and η in (2), and (3), but in no way can eliminate deviations from duality in the sense I explained above. In general, the indices κ and η are model-dependent. Why?

By definition, one can*not* build an exhaustive *theory* of the duality violations based on the first principles. Indeed, assuming there is a certain dynamical mechanism (which goes beyond perturbation theory and condensates) for which such a theory exists, one would include the corresponding component in the theoretical calculation. The reference quantity, $D(Q^2)_{\text{theor}}$, will be redefined accordingly. After the analytic continuation to Minkowski, this will lead, in turn, to a new theoretical spectral density to be used as a reference $\rho(s)_{\text{theor}}$ in the duality relation.

Thus, by the very nature of the problem, it is bound to be treated in models of various degrees of fundamentality and reliability. This is because the duality violation parametrizes our ignorance. Ideally, the models one should aim at must have a clear-cut physical interpretation, and must be tested, in their key features, against experimental data. This will guarantee a certain degree of confidence when these models are applied to the estimates of the duality violations in the processes and kinematical conditions where they had not been tested.

It is rather discouraging that there was very little progress, if at all, in this direction in the last 10 years.

4. AdS/QCD vs. SVZ sum rules

It all started in 1998 when Maldacena; Gubser, Klebanov and Polyakov; and Witten argued [15] that certain four-dimensional super-Yang–Mills theories at large N could be viewed as holographic images of higher-dimensional string theory. In the limit of a large 't Hooft coupling the latter was shown to reduce to anti-de-Sitter supergravity. The framework got the name "Anti-de-Sitter/Conformal Field Theory (AdS/CFT) correspondence."

By now, it is generally believed that ten-dimensional string theory in suitable space-time backgrounds can have a dual, holographic description in terms of superconformal gauge field theories in four dimensions. [2] Conceptually, the idea of a string-gauge duality ascends to 't Hooft [16], who realized that the perturbative expansion of SU(N) gauge field theories in the large-N limit (with the 't Hooft coupling fixed) can be reinterpreted as a genus expansion of discretized two-dimensional surfaces built from the field theory Feynman diagrams. This expansion resembles the string theory perturbative expansion in the string coupling constant. The AdS/CFT correspondence is a quantitative realization of this idea for four-dimensional gauge theories. In its purest form it identifies the "fundamental type IIB superstring in a ten-dimensional anti-de-Sitter space-time background $AdS_5 \times S^5$ with the maximally supersymmetric $\mathcal{N} = 4$ Yang–Mills theory with gauge group SU(N) in four dimensions." The latter theory is superconformal.

Shortly after [15] a new ambitious goal was set: to get rid of conformality and get as close to actual QCD as possible. There are two lines of though. Chronologically the first was the top-down approach pioneered by Witten; Polchinski and Strassler; Klebanov and

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[1] In mathematical literature they refer to the transformation (5) as the inverse Laplace transform. It was first worked in 1930 by Post [13]. Curiously, some questions that are difficult from the general mathematical standpoint seem to be rather trivial for theoretical physicists. I thank Martin Block and Arkady Vainshtein for this reference. See also [14].

[2] Warning: This gauge-gravity duality has nothing to do with the quark-hadron duality which was discussed 5 minutes ago.

Strassler; Maldacena and Nuñez, and others. Here people try to obtain honest-to-god solutions of the ten-dimensional equations of motion, often in the limit of the large 't Hooft coupling when on the string side of the theory one deals with supergravity limit. The problem is: in many instances these solutions are dual to ... sort of QCD ... kind of QCD ..., rather than QCD as we know it. For instance, Witten's set-up or the Maldacena–Nuñez solution guarantee color confinement but the asymptotically free regime of QCD is not attained.

The Klebanov–Strassler supergravity solution is near AdS$_5$ in the ultraviolet limit, a crucial property for the existence of a dual four-dimensional gauge theory. In the ultraviolet this theory exhibits logarithmic running of the couplings which goes under the name of the *duality cascade*. They start from string theory on a warped deformed conifold and discover a cascade of SU(kM)×SU(($k-1)M$) supersymmetric gauge theories on the other side. As the theory flows to the infrared, k repeatedly changes by unity, see the review paper [17]. In the infrared this theory exhibits a dynamical generation of the scale parameter Λ, which manifests itself in the deformation of the conifold on the string side.

There is a variant of the top-down approach in which the requirement of the exact solution of the supergravity equations is "minimally" relaxed. Confinement is enforced through a crude cut-off of the AdS bulk in the infrared, at z_0, where z is the fifth dimension. This leads to a "wrong" confinement, however. In particular, the Regge trajectories do not come out linear. A few years ago, preparing for a talk [18], I suddenly realized that the meson spectrum obtained in this way identically coincides with the 30-year-old result [19] of Alexander Migdal, who, sure enough, had no thoughts of supergravity in five dimensions. His idea was to approximate logarithms of perturbation theory by an infinite sum of poles in the "best possible way." Then this strategy was abandoned since it contradicts OPE. Now, in essence, the Migdal program is resurrected in a new incarnation which goes under the name of AdS/QCD. What was Migdal's goal? He asked himself: "what is the best possible accuracy to which one-loop log Q^2 can be approximated by an infinite sum of infinitely narrow discrete resonances?" and "What are the corresponding values of the resonance masses and residues?" He answered this question as follows: "the accuracy is exponential at large Q^2 and the resonances must be situated at the zeros of a Bessel function." This is exactly the position of the excited ρ mesons found in the first detailed AdS/QCD work [20].

The reason for the coincidence of the 1977 and

2005 results is fully clear (both of them are admittedly wrong). Abstractly speaking, one could have improved at least some aspects, for instance, if instead of supergravity on the string side we could have kept the the full-blown string theory still adhering to the above hard-wall approximation, we would restore asymptotic linearity of the Regge trajectories at large angular momenta J or excitation numbers n. In this limit vein one could then calculate, in addition, say, the meson decay rates, as was done recently by Sonnenschein and collaborators, who recovered the 1979 Casher–Neuberger–Nussinov [21] quasiclassical formula! However, nobody succeeded so far in obtaining the full spectrum of crucial QCD regularities following this road.

This was the reason for the advent of AdS/QCD which I have just mentioned a couple minutes ago. I should add that the bottom-up AdS/QCD guess-and-trial approach was pioneered by Son and Stephanov [22]. If in the AdS/CFT correspondence (string-gauge duality) the five-dimensional metric $g_{AB}(x_\mu, z)$ is "scientifically" *obtained from the Einstein equations* in five dimensions, in AdS/QCD it is postulated *ad hoc*. From AdS/CFT we get, on "our" side of duality, a gauge theory in four dimensions which is superconformal. To get closer to QCD one must break SUSY, conformality and build in the property of asymptotic freedom. To this end people abandon the five-dimensional Einstein equations altogether and *try to guess* appropriate five-dimensional metric. A spectral quantum-mechanical equation with this conjectured metric gives us the hadronic spectral densities. The guideline in the guessing process is the "marriage" between the holographic representation and OPE-based methods, plus chiral symmetry breaking, plus other known regularities of the hadronic world one can squeeze (with luck) in the construction. The theorist is supposed to make various conjectures *en route*. The target is to build a five-dimensional metric encoding as much information on real QCD as possible, and then, with this metric in hands, get new insights and make new predictions.

It is clear that AdS/QCD and SVZ sum rule method share some common features. It is no surprise that many features of the SVZ expansions were recovered. The guessing of the five-dimensional metric is similar to developing the continuum model in the sum rules. Sum rule predictions for low-lying states are rather insensitive to the continuum model. This is an advantage. On the other hand, choosing the five-dimensional metric one completely specifies the relationship between the low-lying states and all higher excitations, with absolute sensitivity to the entire spectrum. Some view this as an advantage, others as disadvantage. For in-

stance, the original hard cut-off metric [20] gives resonances at the zeros of the Bessel function (remember Migdal!). It gives parabolic Regge trajectories, and $\Pi(Q) \sim \ln Q^2 + \exp(-Q)$ in Euclidean (remember Migdal!). It was later replaced by a soft cut-off metric which gives equidistancy in M^2, linear Regge trajectories and rigid $1/Q^2$ corrections in the Euclidean expansion, as if all condensates were expressed in terms of the lowest-order condensate.

By and large, I cannot say that at present AdS/QCD gives a better (or more insightful) description of the hadronic world, than the good old SVZ condensate-based method. Given a rather crude character of the hard-wall and similar approximations, perhaps, today one may hope to extract only universal information on hadronic dynamics, steering clear of all details. However, I do not exclude that further studies of these two approaches, in conjunction, will be beneficial for both and will add some significant understanding to our knowledge of the hadronic world. I appeal to young theorists involved in this area of research to invest effort in this subject.

5. Can holography help?

As it should be clear, I do not expect for the time being the AdS/QCD to produce detailed predictions superior (or even close) in their reliability and precision to those of the SVZ sum rules. However, I hasten to make a reservation. If one can find such general problems whose solution does not depend on particular choices of the five-dimensional metric, one has much better chances to obtain a successful prediction. Such an attempt has been undertaken recently in [23, 24] in a problem in a way related to the chiral anomaly.[3] As we will see, no breakthrough occurred, although the results obtained in [23, 24] give some food for further thought.

It is well known that the longitudinal part of the fermion triangle graph is unambiguously and exactly fixed by the chiral anomaly. In essence, this is a topological rather than dynamical quantity. The knowledge of the above longitudinal part gives us the famous formula for $\pi^0 \to \gamma\gamma$. Vainshtein asked himself a question [25] (see also [26, 27]) whether the transverse part of the triangle graphs is also constrained. He demonstrated that the transverse part $w_T(Q^2)$ of the current-current correlator in an infinitesimally weak electromagnetic

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[3]That was probably the original reason behind the belief that holography will work.

field defined as

$$
\begin{aligned}
\langle j_\mu j_\nu^5 \rangle_{\hat{F}} =& -\frac{1}{4\pi^2}\Big[w_T(q^2)\big(-q^2 \tilde{F}_{\mu\nu} + q_\mu q^\sigma \tilde{F}_{\sigma\nu} \\
& - q_\nu q^\sigma \tilde{F}_{\sigma\mu} \big) + w_L(q^2)\, q_\nu q^\sigma \tilde{F}_{\sigma\mu} \Big], \quad (6)
\end{aligned}
$$

is *not* renormalized in perturbative QCD. However, because of the chiral symmetry breaking, the above non-renormalization theorem is not extendable to to nonperturbative effects [25]. Thus, w_L and w_T have different status: the latter quantity is dynamical. Nevertheless, the fact that the α_s series is absent in w_T gives one hope that only some general aspects of QCD are involved in its calculation. Under a simple additional assumption of the pion dominance, Vainshtein obtained the following analytic "prediction" for the vacuum magnetic susceptibility χ introduced in [28]:

$$
\chi = -\frac{N_c}{4\pi^2 f_\pi^2} \,. \tag{7}
$$

Here N_c is the number of colors, and $f_\pi \approx 92$ MeV is the pion constant. I put "prediction" in the quotation marks because there was no theoretical justification for the above-mentioned simplest assumption, as was *certainly* noted and emphasized in the original paper [25]. I guess, the goal of the holography explorers in this issue was to find a justification for this or a similar relation, perfect the coefficients, and, in general, elevate its status to the level where the quotation marks could be dropped.

First, it was realized [23] that the gravity dual in the case at hand must be Yang–Mills–Chern–Simons theory. Addition of the Chern–Simons term turned out to be absolutely necessary. In the so-called hard-wall version of holography the vacuum magnetic susceptibility was found (numerically) [23] to be close to (7), with the coefficient larger than $N_c/4\pi^2$ by about 10%.

Then other versions of holography, such as the so-called soft model of the "bottom-up" AdS/QCD [22] and the "top-down" Sakai–Sugimoto model [29] (both are popular in this range of questions) were explored in [24]. In this rather broad class of Yang–Mills–Chern–Simons dual theories, with the chiral symmetry broken by the boundary conditions in the infrared, independently of the choice of the gravity metrics, the following relation takes place [24]:

$$
w_T(Q^2) = \frac{N_c}{Q^2} - \frac{N_c}{f_\pi^2}\Big[\Pi_A(Q^2) - \Pi_V(Q^2)\Big]_F, \tag{8}
$$

for any Q^2. Here $\Pi_{A,V}$ are the two-point functions of the axial-vector (vector) currents in the background (very soft) electromagnetic field. The first term in (8) was obtained by Vainshtein. The second term obviously vanishes to any finite order in perturbation theory. This

is a nonperturbative correction found through holography but independent of the particulars of the five-dimensional metric. Equation (8) implies a new set of relations for various resonance parameters.

Considering Eq. (8) at *large Q^2*, using the SVZ-type operator product expansion for $\left[\Pi_A(Q^2) - \Pi_V(Q^2)\right]_F$, and factorization for the four-quark matrix element (justified by the large-N_c limit) one can derive from (8) a consistency condition [24] for the vacuum magnetic susceptibility, in an analytic form. Remarkably, this is exactly the same formula (7) obtained by Vainshtein.

Another example of predictions [24] for physical observables following from (8) are the sum rules

$$\sum_j \frac{g_{\gamma V_i A_j} g_{A_j}}{m_{A_j}^2 - m_{V_i}^2} = -\frac{N_c}{4\pi^2 f_\pi^2} g_{V_i}, \tag{9}$$

$$\sum_i \frac{g_{\gamma V_i A_j} g_{V_i}}{m_{A_j}^2 - m_{V_i}^2} = -\frac{N_c}{4\pi^2 f_\pi^2} g_{A_j}. \tag{10}$$

In the first sum rule i is fixed (and arbitrary) while j runs over all axial-vector resonances. Likewise, in the second expression j is fixed while i runs over all vector resonances. For broad resonances one can substitute the sums by the integrals in the spirit of the SVZ method.

Let us ask ourselves how successful numerically is the Vainshtein formula. The pion constant in (7) is normalized as $f_\pi \approx 92$ MeV. Substituting this number we arrive at $\chi \approx -9.0$ GeV^{-2}. At the same time, the magnetic susceptibility had been determined from the QCD sum rules long ago. Unfortunately, calculation of the magnetic susceptibility, presented in the main text of [28] is not quite correct since an inappropriate dispersion relation was used. The correct result is quoted in 'Note added in proof' in the same paper. The most precise evaluation of the magnetic susceptibility can be found in Sec. 6.3 of the book [30],

$$\chi_{\text{QCD SR}} = -3.15 \pm 0.3 \text{ GeV}^{-2} \tag{11}$$

for the normalization point around 1 GeV. The discrepancy is about a factor of 3, somewhat larger than could have been anticipated.

The main problem with holography which clearly reveals itself in confronting (7) and (11) is that holography, as we know it now, does not reproduce those several terms of OPE which are solidly established. Therefore, fine details of the hadronic picture obtained from holography come out wrong (at least, for the time being), although some overall contours are, perhaps, captured right.

6. Instead of Conclusions

The richness of the hadronic world is enormous.

It describes a very wide range of natural phenomena, e.g.:
- all of nuclear physics;
- Regge behavior and Regge trajectories (highly excited meson and baryon states);
- strongly coupled quark-gluon plasma; high-T phenomena; color superconductivity at high density (through color-flavor locking); neutron stars;
- richness of the hadronic world (chiral phenomena, light and heavy quarkonia, the Zweig rule, glueballs and exotics, exclusive and inclusive processes);
- hadronization of fast moving colored sources, i.e. jets (of special interest are, of course, nonperturbative aspects of the jet physics);
- interplay between strong and weak interactions (in particular, the so-called penguin mechanism);

and many other issues.

At short distances QCD is weakly coupled, allowing high precision perturbative (multi-loop, multi-leg) calculations. However, the advent of the era of arbitrarily exact analytical computations at all energies and momenta, especially in the Minkowski domain, is not expected in the foreseeable future, due to strong coupling nature of the large-distance dynamics. Let us ask ourselves: what do we want from this theory? Is it reasonable to expect high-precision analytic predictions for all low-energy observables? Can we (and should we) compute hadronic masses, matrix elements or, say, proton's magnetic moment up to five digits?

Unlike QED, most probably we will never be able to analytically calculate the above-mentioned and similar observables with this precision. But do we really need this? To my mind, what is really needed is the completion of the overall qualitative picture of confinement + development of various reliable approximate techniques custom-designed for specific applications. The original sum rule method, extended by numerous later developments, fits very well. In this context QCD sum rules do have a future, don't they?

Note Added in December

A very recent publication [31] admired me by its elegance. Zohar Komargodski implemented, in a brilliant way, the old idea [32] that all vector mesons of QCD (i.e. ρ and its excitations) are in fact Higgsed gauge bosons of a hidden non-Abelian local symmetry (or symmetries) of the hadronic world. This was

done in a fully controllable setting of supersymmetric QCD and is heavily based on Seiberg's duality [33]. The dream of Yang and Mills, who originally designed the Yang–MIlls theory in the context of the description of the hadronic world, is realized! The same idea [32] that served as an important catalyst in the advent of AdS/QCD [22], in its supersymmetric reincarnation provided an analytic and parametric proof of the vector meson dominance, a phenomenon that is a crucial feature in the QCD sum rules (albeit seen there numerically rather than parametrically). This clearly tells us that the tool kit available to us for dealing with mysteries of QCD is expanding, and still unsolved mysteries still have chances to be solved in the future.

Acknowledgments

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